



Lecture 1a: Review of The Wave Equation in Dielectric Media

Notation

- MKS units
- Lower case for time varying quantities
- Capitals for the amplitudes of time varying quantities
- Complex quantities used to represent amplitude and phase:

$$a(t) = \text{Re}[Ae^{i\omega t}]$$

- (at least in Chapter 1. In Chapter 2, $E(x,y,z,t) = \text{Re}[E(x,y,z)e^{i\omega t}]$)

Maxwell's Equations



$$\nabla \times \vec{h} = i + \frac{\partial \vec{d}}{\partial t}$$

$$\nabla \times \vec{e} = -\frac{\partial \vec{b}}{\partial t}$$

$$\nabla \cdot \vec{d} = 0$$

$$\nabla \cdot \vec{b} = 0$$

where $\vec{e}$ and $\vec{h}$ are the electric and magnetic field vectors
 $\vec{d}$ and $\vec{b}$ are the electric and magnetic displacement vectors
No free charge.

Constitutive Relations



$$\vec{d} = \epsilon_0 \vec{e} + \vec{p}$$

$$\vec{b} = \mu_0 (\vec{h} + \vec{m})$$

$\vec{p}$ and $\vec{m}$ are the electric and magnetic polarizations of the medium
 ϵ_0 and μ_0 are the electric and magnetic permeabilities of vacuum
 $\vec{e}$ and $\vec{h}$ are the electric and magnetic field vectors
 $\vec{d}$ and $\vec{b}$ are the electric and magnetic displacement vectors

Electric Susceptibility χ (Isotropic)



Isotropic Media: χ is a complex number

$$P = \epsilon_0 \chi E$$

The real part determines the index (velocity) and the imaginary part determines the gain or absorption.

Isotropic media: Vacuum, gasses, glasses (optical fibers)

Anisotropic media: Semiconductors, crystalline materials.

Electric Susceptibility χ (Anisotropic media)

Anisotropic Media: χ is a complex second rank tensor

$$\vec{P} = \epsilon_0 \vec{\chi} \vec{E}$$

$$P_i = \epsilon_0 \sum \chi_{ij} E_j$$

$$P_x = \epsilon_0 (\chi_{xx} E_x + \chi_{xy} E_y + \chi_{xz} E_z)$$

One can always choose a coordinate system such that off axis elements are zero. These are the principal dielectric axes of the crystal. We will only use the principal coordinate system.

$$P_x = \epsilon_0 \chi_{11} E_x$$

$$P_y = \epsilon_0 \chi_{22} E_y$$

$$P_z = \epsilon_0 \chi_{33} E_z$$

Principal Axes



D, E and P are not parallel in general. D and E are related by the electric permeability tensor ϵ

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$$\vec{D} = \epsilon \vec{E}$$

Principal axes can always be chosen such that D and E are parallel and the off diagonal elements of ϵ are zero.

$$\epsilon_{11} = \epsilon_0 (1 + \chi_{11})$$

$$\epsilon_{22} = \epsilon_0 (1 + \chi_{22})$$

$$\epsilon_{33} = \epsilon_0 (1 + \chi_{33})$$

Wave Propagation in Lossless, Isotropic Media

- Lossless: $\sigma=0$, χ is real, ϵ is real.
- Isotropic: χ , ϵ are scalars (not tensors).

$$\nabla \times \vec{e} = i + \frac{\partial \vec{b}}{\partial t} = 0 + \mu \frac{\partial \vec{h}}{\partial t}$$

$$\nabla \times \vec{h} = i + \frac{\partial \vec{d}}{\partial t}$$

$$\nabla \times (\nabla \times \vec{e}) = \mu \frac{\partial (\nabla \times \vec{h})}{\partial t} = \mu \frac{\partial^2 \vec{d}}{\partial^2 t} = \mu \epsilon \frac{\partial^2 \vec{e}}{\partial^2 t}$$

$$\nabla \times (\nabla \times \vec{e}) = \nabla^2 \vec{e} - \nabla (\nabla \cdot \vec{e})$$

$$\nabla^2 \vec{e} = \mu \epsilon \frac{\partial^2 \vec{e}}{\partial^2 t} \quad \text{Wave Equation}$$

Wave Equation



$$e(x, y, z, t) = \text{Re}[E(x, y, z)e^{i\omega t}]$$

$$\nabla^2 \vec{E} + \omega^2 \mu \epsilon \vec{E} = 0$$

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0$$

where

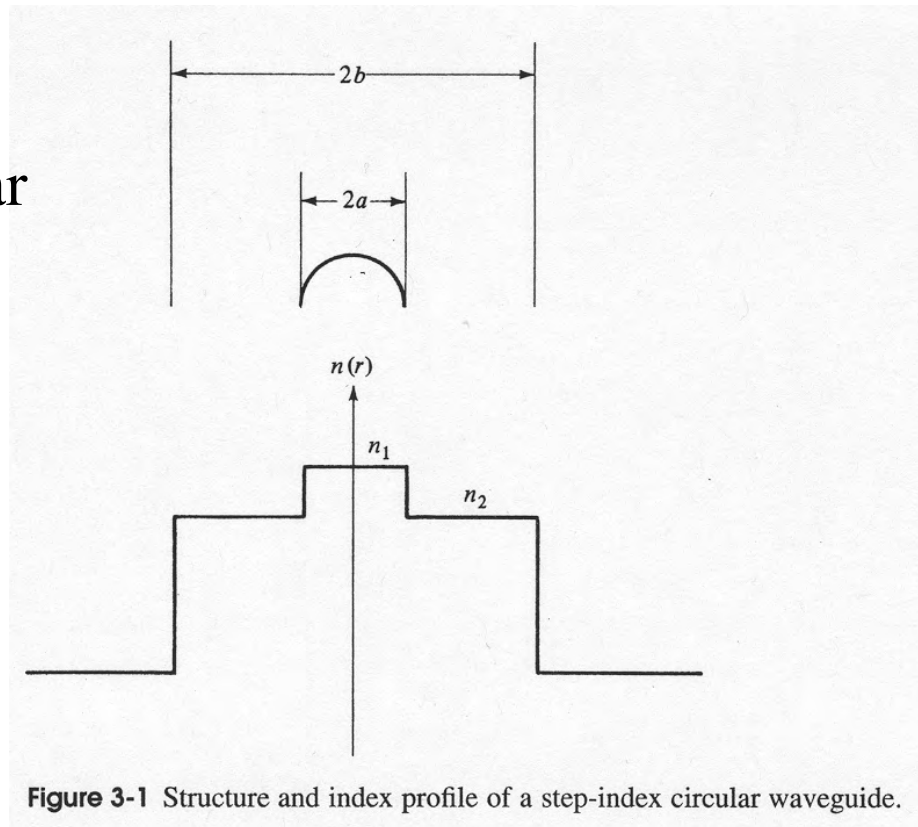
$$k = \omega \sqrt{\mu \epsilon} = \omega n / c$$

$$c = 1 / \sqrt{\mu_0 \epsilon_0}$$

$$n = \sqrt{\frac{\mu \epsilon}{\mu_0 \epsilon_0}}$$

Step Index Circular Waveguide (lossless, isotropic)

- Simplest type of fiber
- (Most fiber these days is far more complex)
- Cylindrical symmetry



Step Index Circular Waveguide (lossless, isotropic)

- Simplest type of fiber
- (Most fiber these days is far more complex)
- Cylindrical symmetry
- Express Laplacian operator in cylindrical coordinates

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Separate variables

$$E_r = \psi(r) \Phi(\phi) e^{i(\omega t - \beta z)}$$

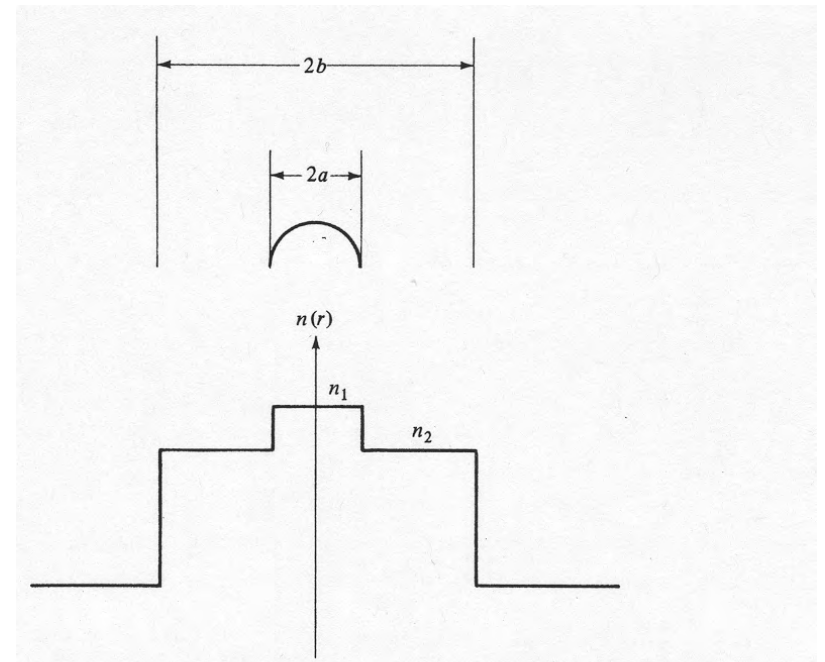


Figure 3-1 Structure and index profile of a step-index circular waveguide.

Separable Solutions



$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + (k^2 - \beta^2) \right] E_z = 0$$

$$E_r = \psi(r) \Phi(\phi) e^{i(\omega t - \beta z)}$$

$$\Phi(\phi) = e^{\pm i l \phi} \quad \text{where} \quad l = 0, 1, 2, \dots$$

Separable Solutions

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + (k^2 - \beta^2) \right] E_z = 0$$

$$E_r = \psi(r) \Phi(\phi) e^{i(\omega t - \beta z)}$$

$$\Phi(\phi) = e^{\pm i l \phi} \quad \text{where} \quad l = 0, 1, 2, \dots$$

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \left(k^2 - \beta^2 - \frac{l^2}{r^2} \right) \right] \psi = 0$$

Bessel differential equation

$$\psi = c_1 J_l(hr) + c_2 Y_l(hr) \quad k^2 - \beta^2 = h^2 > 0$$

$$\psi = c_1 I_l(qr) + c_2 K_l(qr) \quad k^2 - \beta^2 = -q^2 > 0$$

J Bessel function of the first kind

Y Bessel function of the second kind

I Modified Bessel function of the first kind

K Modified Bessel function of the second kind

Boundary Conditions

Decaying fields for $r > a$
 $q > 0$

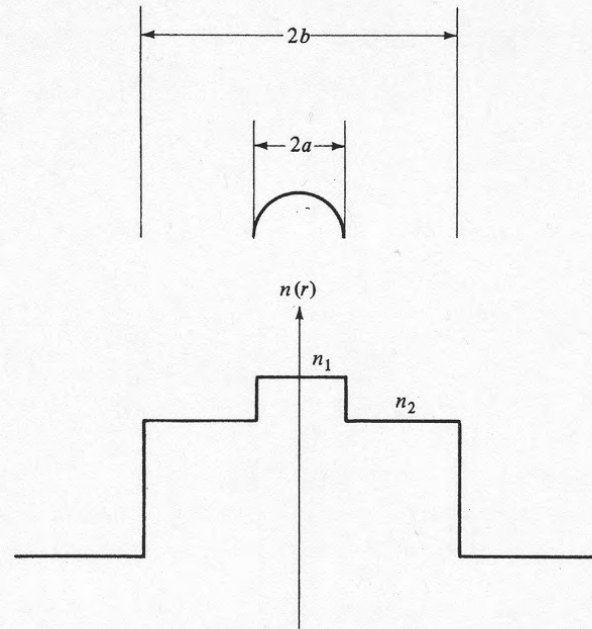


Figure 3-1 Structure and index profile of a step-index circular waveguide.

For fields in the core $r < a$, we need finite fields
(which eliminates Y and K which go to infinity as r approaches 0).

TE $l=0$ Modes

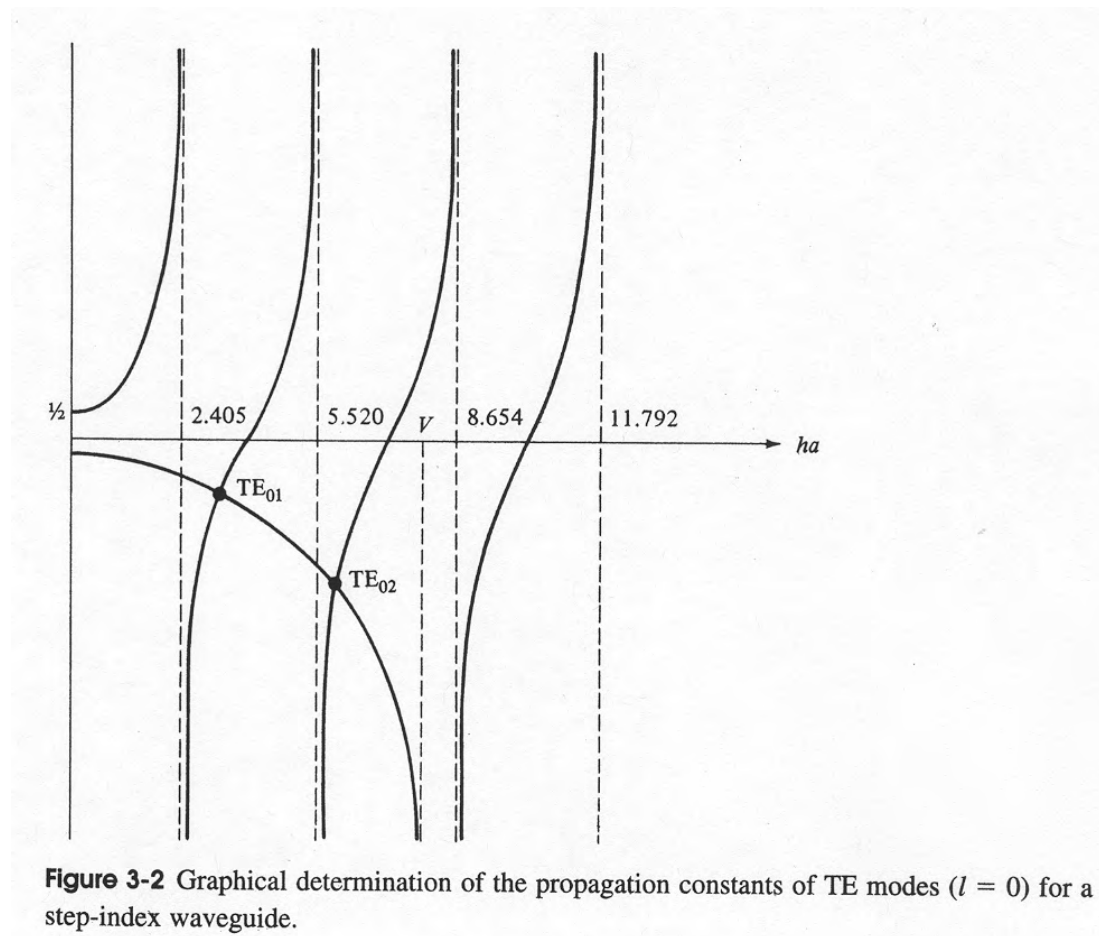


Figure 3-2 Graphical determination of the propagation constants of TE modes ($l = 0$) for a step-index waveguide.

$l=1$ (not TE or TM, but EH)

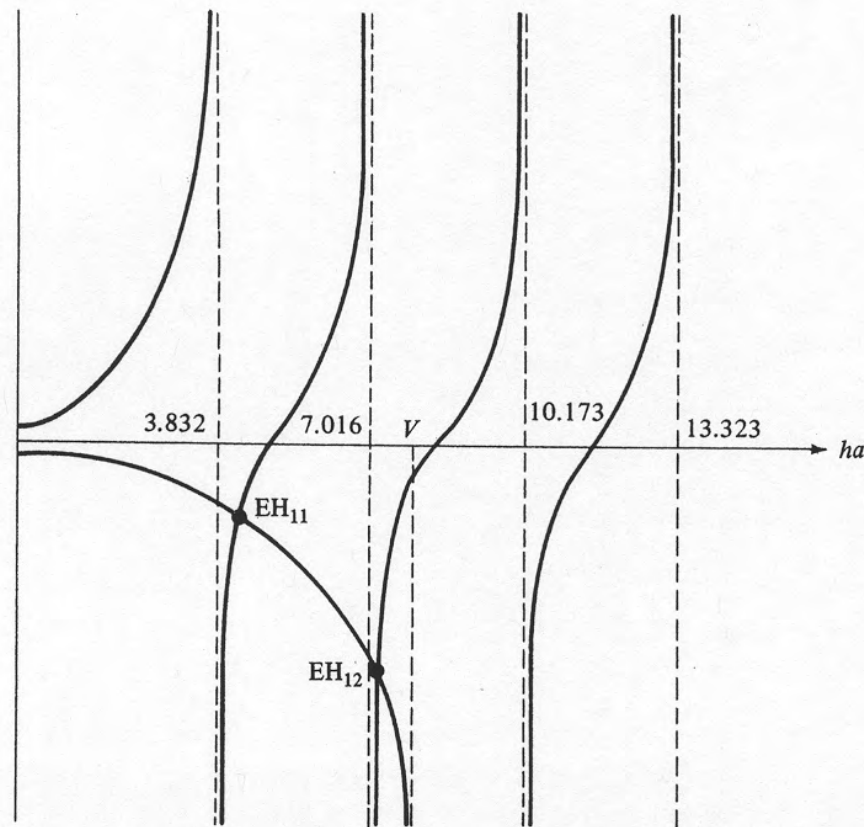


Figure 3-3 Graphical determination of the propagation constants of $l = 1$ EH modes for a step-index fiber.

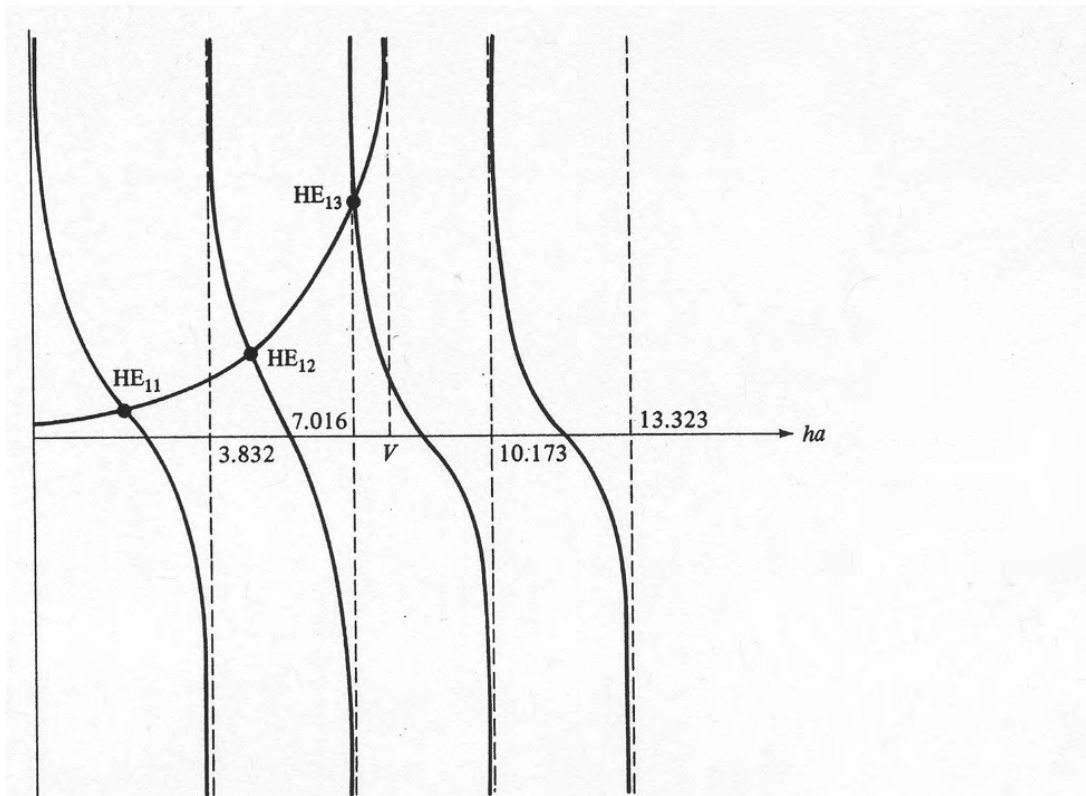


Figure 3-4 Graphical determination of the propagation constants of the $l = 1$ HE modes for a step-index dielectric waveguide.

V parameter

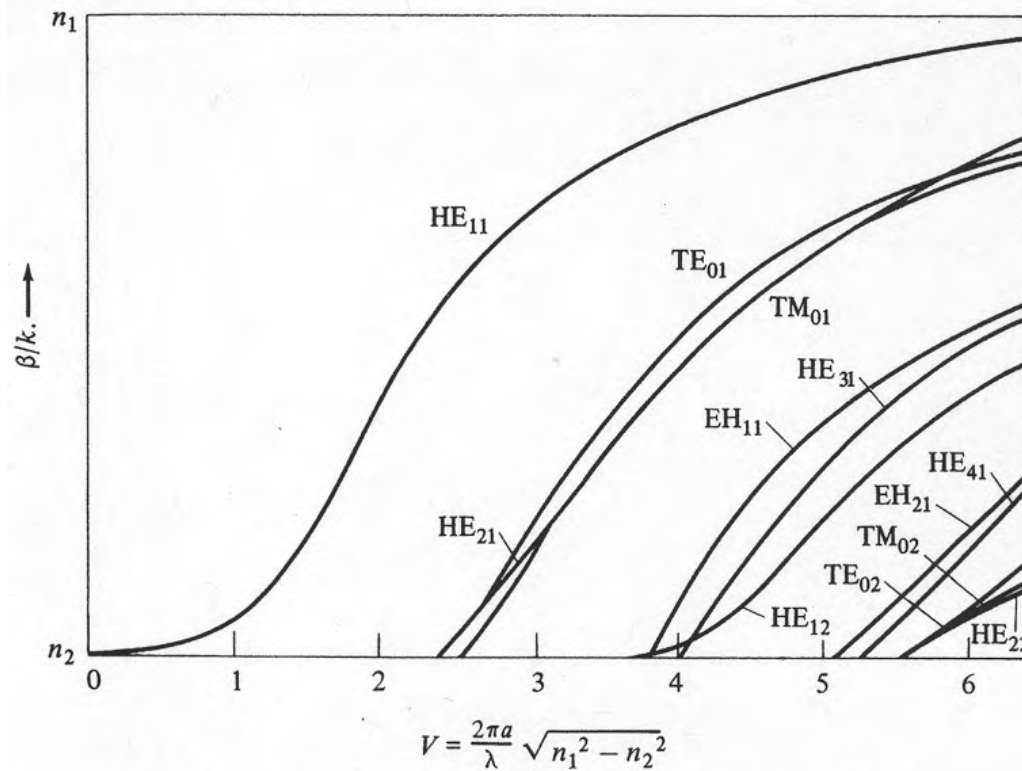


Figure 3-5 Normalized propagation constant as a function of V parameter for a few of the lowest-order modes of a step-index waveguide [4].

For $n_1 - n_2 \ll n_1$, LP approximation is valid.

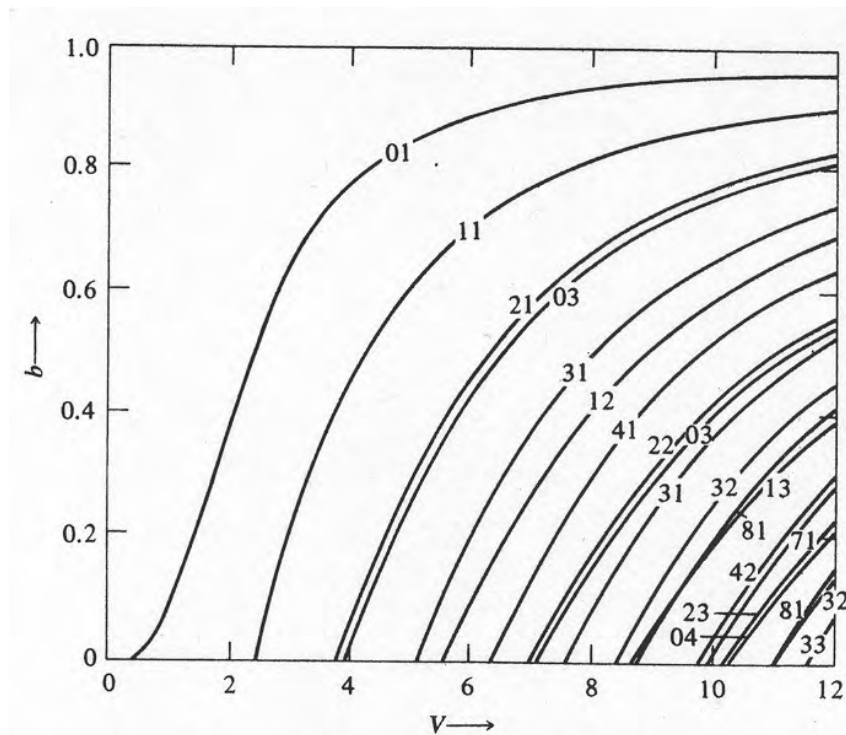


Figure 3-6 Normalized propagation constant b as function of normalized frequency V for the guided modes of the optical fiber, $b = (\beta/k_c - n_2)/(n_1 - n_2)$. (After Reference [5].)

Single mode cut off: $V=2.405$

Degenerate Modes LP_{11}

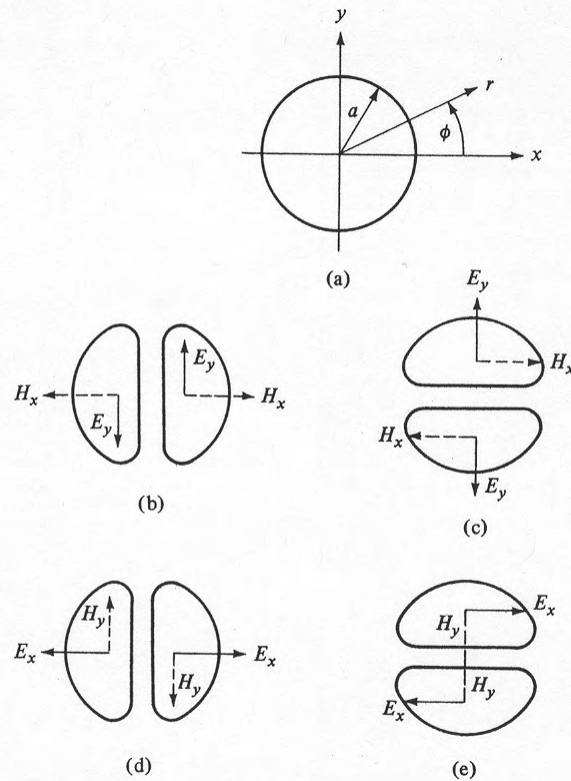


Figure 3-8 Sketch of the fiber cross section and the four possible distributions of LP_{11} .

Modes as a function of V parameter

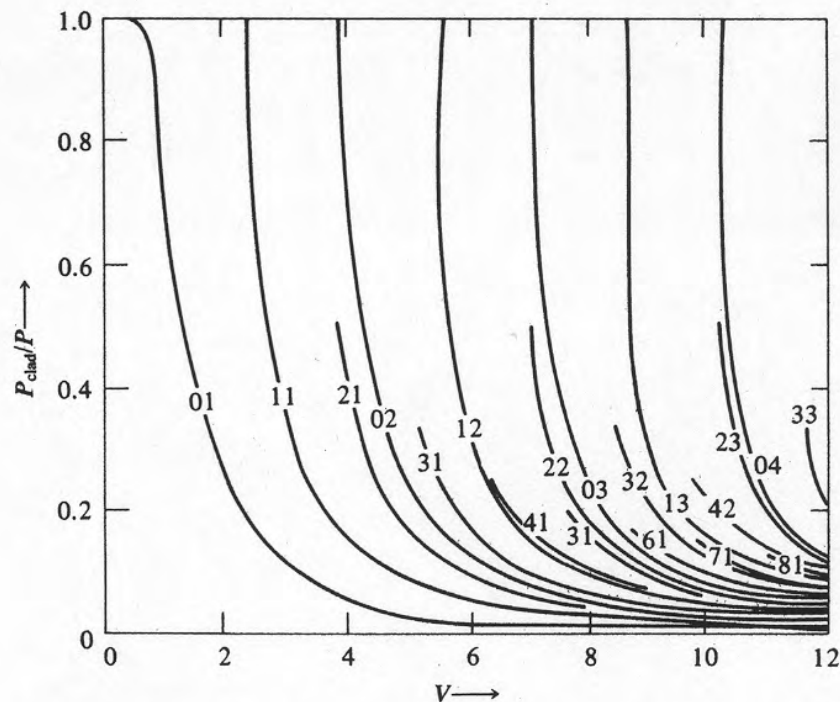
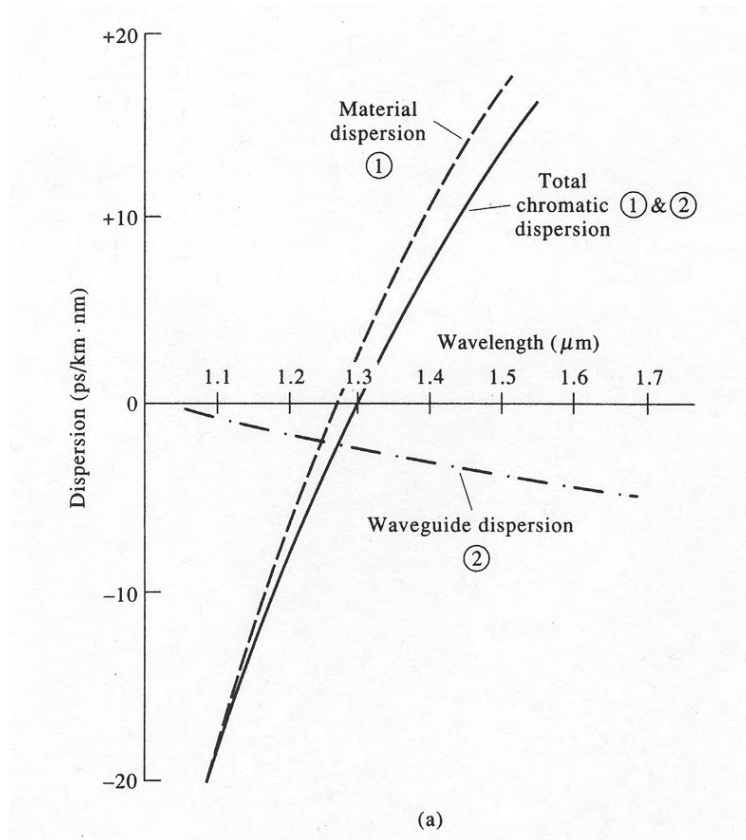


Figure 3-9 Fractional power contained in the cladding as a function of the frequency parameter V . (After Reference [5].)

At cutoff, all the power is in the cladding.

Dispersion



$$\tau = DL\sigma$$

D is dispersion parameter
L is the propagation length
 σ is the spectral width

Dispersion (sum of material and waveguide dispersion)

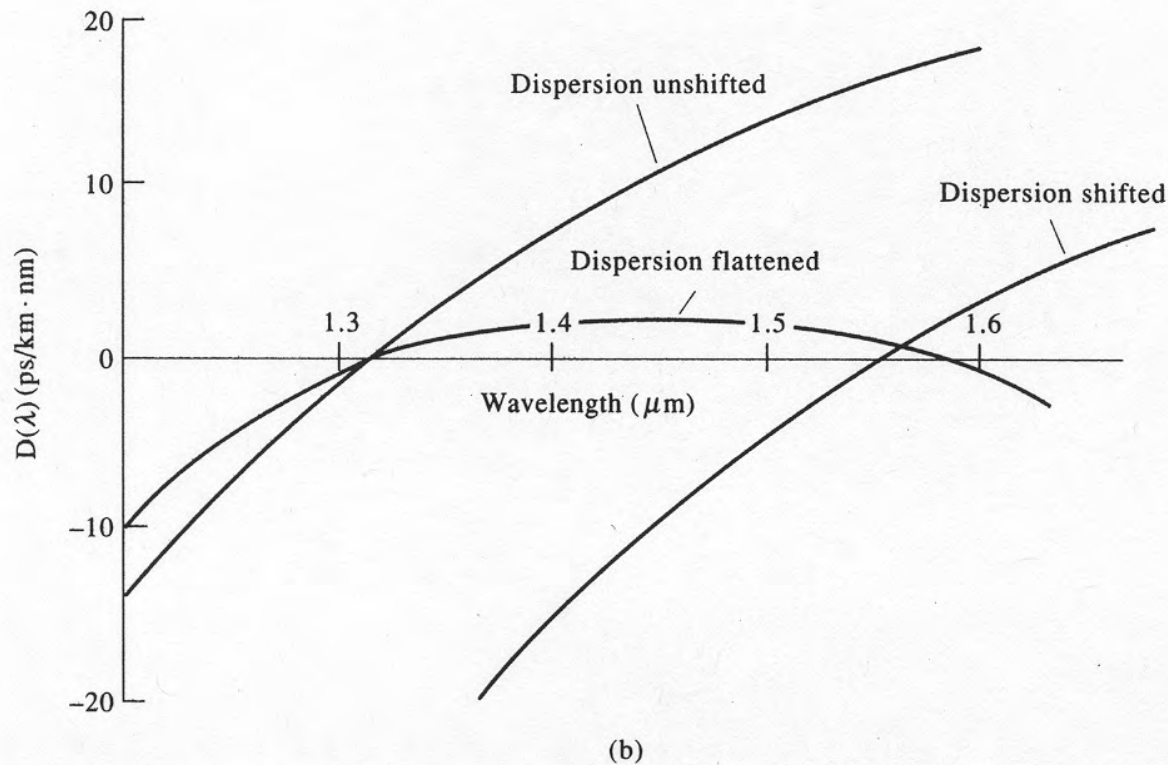


Figure 3-10 Group velocity dispersion of (a) dispersion-unshifted 1.3 μm fiber and (b) dispersion-flattened and dispersion-shifted fibers. (After Reference [1].)

Loss in early optical fibers

(now the O-H peaks around $1.4 \mu\text{m}$ are small)

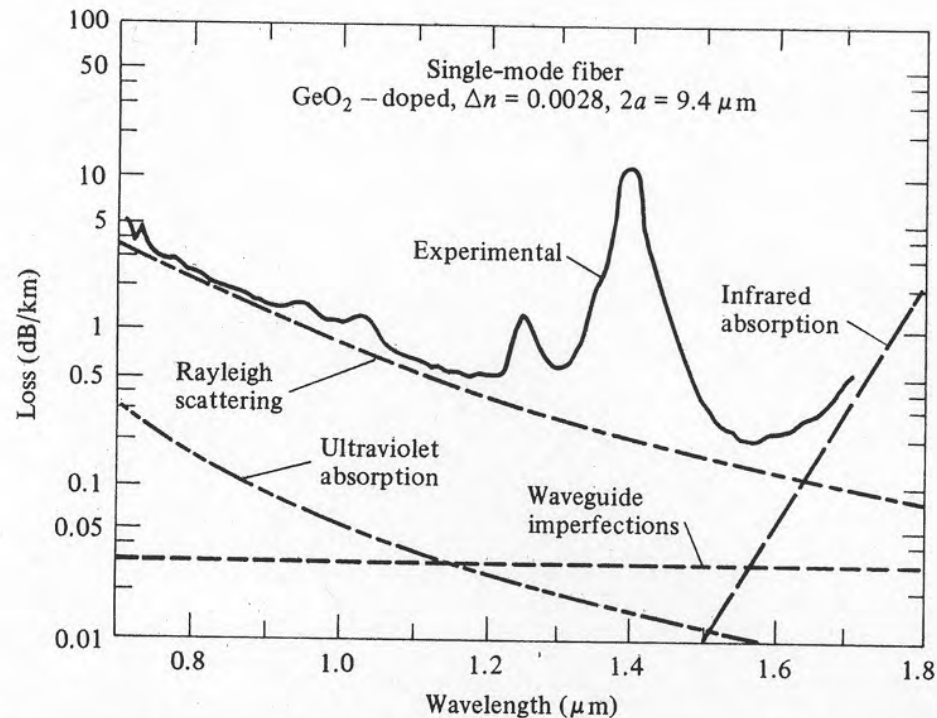


Figure 3-19 Observed loss spectrum of a germanosilicate single-mode fiber. Estimated loss spectra for various intrinsic materials effects and waveguide imperfections are also shown. (From Reference [20].)

Summary



- Single mode condition required for high performance
- Multimode fiber used for low cost
- Dispersion is designable.
- 1.3 micron: zero of dispersion
- 1.55 micron: minimum loss
- Zero dispersion is not good because of nonlinearity